On Integrating Out Heavy Fields in SUSY Theories

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Abstract

We examine the procedure for integrating out heavy fields in supersymmetric (both global and local) theories. We find that the usual conditions need to be modified in general and we discuss the restrictions under which they are valid. These issues are relevant for recent work in string compactification with fluxes.

PACS numbers: 11.25. -w, 98.80.-k; COLO-HEP 508

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In a theory containing both heavy (mass M) and light (mass $m \ll M$) fields, one may derive an effective field theory [1] valid for energy scales $E \ll M$, by doing the functional integral over the heavy fields (as well as light field modes with frequencies greater than M). If one is just considering the classical approximation then this just means solving the classical equation for the heavy field in terms of the light field and substituting back into the action. So if the potential for the heavy (Φ) and light (ϕ) fields is $V(\Phi,\phi) = \frac{1}{2}M^2\Phi^2 + \tilde{V}(\Phi,\phi)$ the equation of motion for the heavy field gives,

$$\Phi = \frac{1}{\Box - M^2}\frac{\partial\tilde{V}}{\partial\Phi} = -\frac{1}{M^2}\frac{\partial\tilde{V}}{\partial\Phi} + \frac{1}{M^2}O(\frac{\Box}{M^2}\frac{\partial\tilde{V}}{\partial\Phi}).$$

In other words up to terms in the derivative expansion that maybe ignored at energy $E \ll M$, Φ is a solution of the equation

$$\frac{\partial V}{\partial\Phi} = 0 \tag{1}$$

Consider now a globally supersymmetric theory of chiral scalar fields with superspace action

$$S = \int d^4xd^4\theta\bar{\Phi}^i\Phi^i + (\int d^4xd^2\theta W(\Phi^i) + c.c.]. \tag{2}$$

The superfield equations of motion are (with $D^2 = D^\alpha D_\alpha$where D_α is the spinor covariant derivative)[6],

$$-\frac{1}{4}\bar{D}^2\bar{\Phi}^i + \frac{\partial W}{\partial\Phi^i} = 0. \tag{3}$$

The component form of this is obtained by successive application of the spinorial covariant derivative and setting $\theta = \bar{\theta} = 0$ (indicated below by the symbol $|$). In particular the auxiliary field equation (from D^0) and the scalar field (from $-\frac{1}{4}D^2$) become (after setting the fermions to zero and using $D^2\bar{D}^2\bar{\Phi} = 16\Box\bar{\Phi}$)

$$\bar{F}^i + \frac{\partial W}{\partial\Phi^i}| \ = 0 \tag{4}$$

$$\Box\bar{\Phi^i}| + \sum_j \frac{\partial W}{\partial\Phi^i\partial\Phi^j}|F^j \ = 0 \tag{5}$$

Now suppose that we wish integrate out a heavy field with say $i = H$ to get an effective theory for the light fields with $i = l$. For example one might have $W = \frac{1}{2}M\Phi^{H2} + \frac{1}{2}\lambda_{ll'}\Phi^H\Phi_l\Phi_{l'} + W_L(\Phi_l)$. Then the analog of (1) (see for example [2]) is to require that

$$\frac{\partial W}{\partial\Phi^H} = 0. \tag{6}$$

Taking components of this equation, (and ignoring the fermionic terms)

$$\frac{\partial W}{\partial \Phi^H}| = 0 \tag{7}$$

$$\sum_j \frac{\partial^2 W}{\partial \Phi^H \partial \Phi^l}| D^2 \Phi^l + \frac{\partial^2 W}{\partial \Phi^H \partial \Phi^H}| \frac{\partial \bar{W}}{\partial \bar{\Phi}^H}| = 0 \tag{8}$$

where in the second (obtained by acting with D^2 and setting fermions to zero) we have used (4) for $i = H$. The second condition is of course the requirement that the potential $V = \sum_i F^i \bar{F}^i$ be extremized with respect to Φ_H, which is what one would impose in a non-supersymmetric theory (see (1)). However here we have the additional condition (7) which in conjunction with (8) and the equatinon of motion for the light field leads to

$$\sum_j \frac{\partial^2 W}{\partial \Phi^H \partial \Phi^l}| \frac{\partial W}{\partial \Phi^l}| = 0$$

But (after solving for the heavy field using (6) this is a constraint on the light fields in theories where there is a non-zero coupling between the heavy and light fields as in the above example. In particular if there is only one light field it imposes the condition $\frac{\partial W}{\partial \Phi^l}| = 0$, meaning that the light scalar is also at the minimum of the potential. In other words one does not get an effective potential for the light field - the only consistent result of integrating out the heavy field is that all fields are sitting at the SUSY minimum (if it exists). When there is more than one light field this is not necessarily the case but nevertheless the light field space is constrained. If we had kept the fermionic terms then this constraint would be a relation between the bosonic components and squares of fermionic components of the light fields.

To see where this comes from let us write the superpotential for the heavy light theory as

$$W(\Phi^H, \Phi^l) = \frac{1}{2} M \Phi^{H2} + \tilde{W}(\Phi^H, \Phi^l). \tag{9}$$

Operating on the (conjugate of the) equation of motion (3) with $(-\frac{1}{4}D^2)$, using (3) again and rearranging we have,

$$\Phi^H = \frac{1}{\Box - M^2} \left(M \frac{\partial \tilde{W}}{\partial \Phi^H} + \frac{\bar{D}^2}{4} \frac{\partial \bar{\tilde{W}}}{\partial \bar{\Phi}^H}\right)$$

Expanding the inverse Klein-Gordon operator as before we can rewrite this as,

$$M \Phi^H + \frac{\partial \tilde{W}}{\partial \Phi^H} = -\frac{\bar{D}^2}{4M} \frac{\partial \bar{\tilde{W}}}{\partial \bar{\Phi}^H} + O\left(\frac{\Box}{M^2}(...)\right).$$

So to the lowest order in the space-time derivative (momentum) expansion what we get for the equation determining the heavy field in terms of the light is,

$$\frac{\partial W}{\partial \Phi^H} = -\frac{\bar{D}^2}{4M}\frac{\partial \bar{\tilde{W}}}{\partial \bar{\Phi}^H}, \tag{10}$$

rather than (6). To get the latter one needs the additional assumption that the possible values of $\bar{D}^2 \frac{\partial \bar{\tilde{W}}}{\partial \Phi^H}$ are small compared to M. For instance in the above example (in the paragraph after (5) this means that we need $\Phi_l << M$. This of course is what one would expect. However as we pointed out earlier in this same approximation the light field space appears to be constrained.

Let us be even more specific and consider the model (9) with $\tilde{W} = \frac{1}{2}HL^2$. (We've relabelled the heavy field as H and the light field as L.) Then (10) becomes,

$$\begin{aligned}\frac{\partial W}{\partial H} &= MH + \tfrac{1}{2}L^2 = -\frac{\bar{D}^2}{4M}(\frac{1}{2}\bar{L}^2) \\ &= -\tfrac{1}{4M}(\bar{D}^{\dot{\alpha}}\bar{L}\bar{D}_{\dot{\alpha}}\bar{L} + \bar{L}\bar{D}^2\bar{L})\end{aligned} \tag{11}$$

Again we see that the strict imposition of $\partial W/\partial H = 0$ leads to the constraint on the light field space that we found above. To see in what approximation this equation is valid let us solve it for H (giving $H = -L^2/M$) and then plug it back into the RHS of the last equality of (11) after using the light field equation. The bosonic term is then $L^2\frac{|L|^2}{M^2}$and is small when $|L| << M$.

It is perhaps worthwhile looking at this example in component form. The superspace Lagrangian in the above example is

$$\int d^4\theta(\bar{H}H + \bar{L}L) + \left[\int d^2\theta\frac{1}{2}(MH^2 + HL^2) + c.c.\right] \tag{12}$$

In components one has, writing the scalar and F components of $H = (A,\mathrm{F})$ and of $L = (a, f)$ and ignoring the fermion terms,

$$\begin{aligned}L &= \bar{A}\Box A + \bar{a}\Box a + \bar{F}F + \bar{f}f + \frac{1}{2}(Fa^2 + \bar{F}\bar{a}^2) + (Aaf + \bar{A}\bar{a}\bar{f}) \\ &+ M(AF + \bar{A}\bar{F})\end{aligned} \tag{13}$$

The heavy field equations are

$$-\Box\bar{A} = MF + af \tag{14}$$

$$\bar{F} = -\frac{a^2}{2} - MA \tag{15}$$

Solving them we get after expanding in powers of $\Box/M^2$,

$$\begin{aligned}\bar{A} &= \frac{1}{2M^2}(-M\bar{a}^2 + 2af) + \frac{1}{M^2}O(\frac{\Box}{M^2}) \\ \bar{F} &= -\frac{\bar{a}\bar{f}}{M} + \frac{1}{M^2}O(\frac{\Box}{M^2})\end{aligned}$$

It is easily checked that these are precisely the equations that would be obtained by looking at the components of the superfield equation (10). Plugging these equations into (13) we get the light field potential,

$$-V = \bar{f}f(1 + \frac{\bar{a}a}{M^2}) - \frac{1}{2M}(fa^3 + \bar{f}\bar{a}^3)$$

Eliminating the light auxiliary field we have

$$V = \frac{|a|^6}{4M^2}(1 + \frac{|a|^2}{M^2})^{-1}$$

If we had just imposed the usual condition $\partial W/\partial H = 0$ we would not have got the $|a|^2/M^2$term in the parenthesis. This means that this condition gives the correct result for the potential only for small values of the field $|a| << M$.

To derive the analog of (10) in supergravity we use the formalism in chapter 8 of [3] (though we remain with the conventions of [4]). In this formalism the matter equations can be derived by replacing the supervielbein determinant by the so-called chiral compensator field ϕ and treating the coupling of the matter fields as in flat superspace. Thus the super-covariant derivative $D_\alpha$is the flat space one and satisfies $D^3 = 0$. Also acting on chiral fields we have $(D^2\bar{D}^2/16)\Phi = \Box\Phi$. The action is (with $M_p^2 = 1$)

$$S = -3\int d^4xd^4\theta\bar{\phi}\phi e^{-K/3} + (\int d^4xd^2\theta\phi^3 W + h.c.) \tag{16}$$

where the superpotential is a holomorphic function of the chiral scalar fields $W = W(\chi^i)$ and the Kaehler potential is a real function $K = K(\chi, \bar{\chi})$. From this action one obtains the following equations of motion.

$$\begin{aligned}\frac{1}{4}\bar{D}^2\bar{\chi}^{\bar{k}} + \frac{1}{4}K^{\bar{k}i}(K_{i\bar{j}\bar{l}} - \frac{2}{3}K_{i\bar{j}}K_{\bar{l}})\bar{D}^{\dot{\alpha}}\bar{\chi}^{\bar{j}}\bar{D}_{\dot{\alpha}}\bar{\chi}^{\bar{l}} \ +\tfrac{1}{2}\tfrac{\bar{D}^{\dot{\alpha}}\bar{\phi}}{\bar{\phi}}\bar{D}_{\dot{\alpha}}\bar{\chi}^{\bar{k}} \\ = e^{K/3}\frac{\phi^2}{\bar{\phi}}K^{\bar{k}i}D_iW \\ \frac{1}{4}\bar{D}^2(\bar{\phi}e^{-K/3}) = -\phi^2 W(\chi)\end{aligned}$$

Using the identity above (16) we get,

$$\Box\bar{\chi}^{\bar{k}} = -\frac{D^2}{16}[K^{\bar{k}i}(K_{i\bar{j}l} - \frac{2}{3}K_{i\bar{j}}K_{\bar{l}})]\bar{D}^{\dot{\alpha}}\bar{\chi}^{\bar{j}}\bar{D}_{\dot{\alpha}}\bar{\chi}^{\bar{l}} - \bar{\phi}^{-1}\frac{D^2}{4}(e^{K/3}\phi^2 D^{\bar{k}}W) \tag{17}$$

Let us now consider the case with one heavy superfield H which we will take to be canonically normalized. So the Kahler potential becomes,

$$K = \bar{H}H + K^l(L, \bar{L}) \tag{18}$$

where L stands for the light fields. Also the superpotential is taken to be

$$W = \frac{1}{2}MH^2 + \tilde{W}(H, L)$$

where $M$is a large mass parameter.

Then from (17) we have for the heavy field

$$\begin{aligned}
-4\bar{\phi}\Box\bar{H} &= D^2(e^{K/3M_p^2}\phi^2)D_HW + 4Me^{2K/M_p^2}\phi^2\bar{\phi}^2(1+\tfrac{\bar{H}H}{M_p^2})D_{\bar{H}}\bar{W} \\
+ \quad & (1+\tfrac{\bar{H}H}{M_p^2})[e^{K/3M_p^2}\phi^2((\tfrac{2}{3}\tfrac{K_l}{M_p^2}D^\alpha\chi^l - \tfrac{2D^\alpha\phi}{\phi}) + \tfrac{\bar{H}D^\alpha H}{M_p^2}) \qquad +2D^\alpha(e^{K/3M_p^2}\phi^2)]D_\alpha H \\
- \quad & \tfrac{\bar{\phi}}{6M_p^2}D^2K_{\bar{l}}\bar{D}^{\dot{\alpha}}\bar{H}\bar{D}_{\dot{\alpha}}\bar{\chi}^{\bar{l}} + 2D^\alpha(e^{K/3M_p^2}\phi^2)D_\alpha D_H\tilde{W} \qquad +e^{K/3M_p^2}\phi^2D^2D_H\tilde{W}
\end{aligned} \tag{19}$$

In the above $D_HW = \partial_HW + K_HW/M_p^2$ is the Kaehler derivative of the superpotential with respect to the heavy field and we have restored the dependence on the Planck mass. To integrate out a heavy field H we have to set $\Box\bar{H} \to p^2\bar{H} \to 0$ and the condition for that is that the right hand side of the above equation is set to zero. Note that this condition reduces to (10) in the global limit $M_p \to \infty$ and $\phi \to 1$. Here up to terms involving a factor of $D_\alpha H$ we have

$$D^2(e^{K/3M_p^2}\phi^2)D_HW + 4Me^{2K/M_p^2}\phi^2\bar{\phi}^2(1+\frac{\bar{H}H}{M_p^2})D_{\bar{H}}\bar{W} = -e^{K/3M_p^2}\phi^2D^2D_H\tilde{W}$$

As in the case of the global SUSY discussion one may expect that with some restriction on the light field space (so that the right hand side of the equation is small compared to M) the relevant condition would be the natural generalization of (6)

$$D_HW = \partial_HW + \frac{1}{M_p^2}W\partial_HK = 0 \tag{20}$$

Note that (by taking its spinor derivative) this condition implies

$$W\bar{D}_{\dot{\alpha}}\bar{H} = 0,$$

so that the other $O(M)$ terms which all have a factor of $D_\alpha H$ also vanish (since the superpotential should not vanish at a generic point). So (20) is certainly a sufficient condition in the sense that it implies $\Box H = 0$.

However it is easy to see that the strict implementation of the condition (20) leads to the conclusion that the light fermion fields would have to be set to zero. Let us look at this in somewhat more general terms than above.

We assume that the Kaehler potential is a sum of terms as in the string theory examples discussed in [5]. In particular if we call the heavy superfields H^I and the light superfields L^i assume that

$$K = K^h(H, \bar{H}) + K^l(L, \bar{L}) \tag{21}$$

Thus in the example in section 4 below, $H^I = z^i$and $L^i = S, T$. Then the generalization of (20) becomes (note that capital letters I, J go over the heavy fields)

$$K^{h\bar{I}J} D_J W = 0 \tag{22}$$

Using the non-degeneracy of the metric on the heavy fields this becomes,

$$\partial_I W + K^h_I W = 0 \tag{23}$$

Taking the anti-chiral derivative of this equation and using the chirality of the superpotential we get,

$$W K^h_{I\bar{J}} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{H}^{\bar{J}} = 0 \tag{24}$$

implying $\bar{\mathcal{D}}_{\dot{\alpha}} \bar{H}^{\bar{J}} = 0$ since the metric is $K^h_{I\bar{J}}$ is non-degenerate and $W \neq 0$ at generic points.

Now let us assume that there is a solution $H = H(L, \bar{L})$ of equation (23). Using the chirality of H and L we get by differentiating this solution,

$$\bar{\mathcal{D}}_{\dot{\alpha}} H = \frac{\partial H^I}{\partial L^j} \bar{\mathcal{D}}_{\dot{\alpha}} L^j + \frac{\partial H^I}{\partial \bar{L}^{\bar{j}}} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{L}^{\bar{j}} = \frac{\partial H^I}{\partial \bar{L}^{\bar{j}}} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{L}^{\bar{j}} = 0. \tag{25}$$

Similarly from the result (see (24)) that the chiral derivative of H also vanishes,

$$\mathcal{D}_\alpha H^I = \frac{\partial H^I}{\partial L^j} \mathcal{D}_\alpha L^j + \frac{\partial H}{\partial \bar{L}^{\bar{j}}} \mathcal{D}_\alpha \bar{L}^{\bar{j}} = \frac{\partial H^I}{\partial L^j} \mathcal{D}_\alpha L^j = 0 \tag{26}$$

These two equations tell us that the chiral derivative of L should be zero - in other words the light fermions should also be set to zero. This of course means that the light

field theory is not supersymmetric! However the problem is that the condition (20) or its generalization is really too strong. It was obtained by ignoring the $D_\alpha H$ terms as well as the $O(1/M)$ terms. Of course as was pointed out earlier, these terms are zero if one imposes (20) so that one gets $\Box H = 0$, but the condition itself is not necessary. The actual necessary condition which follows from (19) is a relation between the Kahler derivative terms and the fermionic superfield terms. This condition would then express the bosonic component of the heavy superfield in terms of the light bosonic field as well as squares of light fermionic fields. However if we are interested only in the scalar potential we do not need to keep these terms. Thus the condition (20) can clearly be still used if one is just interested in computing the potential for the light chiral scalars.

Unlike the case of rigid supersymmetry (20) is not a holomorphic equation since the Kaehler derivative involves the real function $K(\Phi, \bar{\Phi})$. This means that the solution for the heavy field will not in general be a holomorphic function of the light fields and hence the light field theory in general will have a superpotential that is just one (or a constant) and the whole effect of integrating out the heavy field will be accounted for by changing the Kaehler potential. In fact the original potential should be expressed in terms of the Kaehler invariant function $G = \ln K + \ln |W|^2$before integrating out the heavy fields. In a companion paper [5] we show this explicitly in some examples that come from type IIB string theory compactifications with fluxes.

Now one might worry that the restriction on the range of the light field essentially forces us back to the global case. This would indeed be the case in an example such as (12). Evaluating (20) in this case we have,

$$D_H W = MH + \frac{1}{2}L^2 + \bar{H}\frac{1}{2}(MH^2 + HL^2) = 0$$

Solving this for H we may compute the scalar potential using the standard supergravity formula (expressed in terms of G) but now the question is whether it is consistent to keep the supergravity corrections given that the integrating out formula above, is valid only for $|a|^2/M^2 << 1$ as in the global case discussed earlier. The point is that necessarily $M \leq M_p$ so that supergravity corrections which in this model are $O(|a|^2/M_p^2)$ should also be ignored for consistency of the approximation.

However very often in string theoretic examples (such as those with flux compactifications) there is a constant in the superpotential $W = W_0 + ...$ where the ellipses denote field

dependent terms. In these cases the supergravity corrections are indeed significant since $D_L W = \partial_L W + K_L W/M_p^2 \simeq \partial_L W + K_L W_0/M_P^2$ and the second term may even be of $(O(1)$ even thought light field space is restricted to $|a|^2 << M^2$. So this does not necessarily force us to the global limit since in many examples of interest in string theory there would be a constant in the superpotential which is generically of the order of the Planck/String scale. Thus the extra piece in the Kaehler derivative (as compared to the ordinary derivative) of the superpotential has to be kept. So we may use the condition (22) with the understanding that it is to be used only for the scalar components of the superfields for the purpose of calculating the light scalar field potential, still remaining within the context of supergravity.

Acknowledgments:

I'm very grateful to Martin Rocek for several useful suggestions.

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[1] S. Weinberg, Phys. Lett. **B91**, 51 (1980).

[2] K. A. Intriligator and N. Seiberg, Nucl. Phys. Proc. Suppl. **45BC**, 1 (1996), hep-th/9509066.

[3] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, Superspace, or one thousand and one lessons in supersymmetry - Front. Phys. **58**, 1 (1983), hep-th/0108200.

[4] J. Wess and J. Bagger, Supersymmetry and supergravity (1992), princeton, USA: Univ. Pr. 259 p.

[5] S. P. de Alwis (2005), hep-th/0506266.

[6] We use the conventions of [4]. For the derivations below see also [3].